

## Reasoning and proving in mathematics teacher education

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*There is evidence for recommendations to link mathematics teacher education (MTE) closely to school mathematics and to emphasise proving why rather than proving that when teaching reasoning and proof (R&P) in schools. In spite of that we suggest not to take the implication that MTE focuses on proving why to extremes. We outline the background, framework, and results of a pilot to an intervention study that seeks to address the problems of R&P in MTE. The results suggest that teachers face more problems with R&P than expected and have difficulties just selecting situations from school in need of a mathematical justification, let alone developing justifications and supporting their students' learning of R&P. This supports our suggestion that a dual emphasis on proving that and proving why is needed in MTE.*

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Recommendations for teacher education increasingly emphasise issues that are specific to the profession. This is so in suggestions that academic mathematics does not suffice as teachers' content preparation (e.g. Ball, Thames, & Phelps, 2008; e.g. Rowland, Turner, Thwaites, & Huckstep, 2009). It is apparent in the explicit emphasis on the tasks of teaching in the college based parts of programmes, which involves a shift “from a focus on what teachers know and believe to a greater focus on what teachers do” (Ball & Forzani, 2009, p. 503). And the professional emphasis shows in suggestions that teaching-learning processes in mathematics teacher education (MTE) should model those envisaged for school mathematics if reform recommendations are to materialize in school (Krainer, 1998; Lunenberg, Korthagen, & Swennen, 2007).

We present the pilot to an intervention study for teachers in primary and lower secondary school in Denmark. The study is on reasoning and proving (R&P), a notoriously difficult topic for all students, including prospective teachers. In line with the general trend outlined above, it has been suggested to use approaches recommended for schools in mathematics teacher education and shift the emphasis from the disciplinary practice of *proving that* to *proving why* (Rowland, 2002). Our study, *Reasoning and Proving in Teacher Education (RaPiTE)*, is in line with this recommendation, but building on studies of practising teachers we argue that it should not be taken too far. In a sense to be explained later, we suggest that prospective teachers need to become engaged in R&P processes that are “sufficiently close” to classroom practice as well as to the discipline of mathematics.

As the rationale of RaPiTE is based on research with practising teachers, the pilot addresses the question of whether the dual emphasis on school and academic mathematics is suitable for MTE. One result of the pilot is that research participants face even greater difficulties with R&P than expected and have problems identifying classroom situations in need of a mathematical justification, let alone devising justifications and engaging their students in working with them. This supports that MTE needs to be close to both instruction in schools and to the disciplinary practice of *proving that*, if teachers are to facilitate their students' proficiency with R&P.

Below we outline recent scholarship on R&P in school mathematics and MTE. We then present our framework, Patterns of Participation (PoP), and elaborate on the approach of RaPiTE, before describing the organisation, methods, and results of the pilot. We finish with a discussion of how the results relate to the main study.

## **R&P IN SCHOOL AND IN MATHEMATICS TEACHER EDUCATION**

It is generally agreed that R&P are treated with less care than they deserve in schools and that mathematical argumentation at times degenerates into authoritative proof schemes (Harel, 2007). Proofs are often dealt with in secondary geometry only and for the purpose of verification of results that are presented ready-made. A.J. Stylianides (2007a) suggests that the late introduction of proofs may cause a disconnect in students' mathematical experiences that contributes to later problems with R&P. Yackel & Hanna (2003) argue that current uses of proof do not allow students to develop understandings of and proficiency with the multiple purposes of proving and do not facilitate their understanding of the contents in question.

Addressing these problems, Yackel and Hanna (2003) focus on the exploratory and communicative functions of proof. NCTM (2008) locates proof in a reasoning-and-proof cycle of exploration, conjecture, and justification. Emphasising proving as a specifically mathematical mode of justification, A.J. Stylianides (2007b) suggests that proving in school (and elsewhere) be understood as making mathematically valid inferences on the basis of what is or may become taken-as-shared in terms of content and modes of argumentation in the community in question. These recommendations seek to engage students at all school levels in a range of R&P-processes, including developing specifically mathematical justifications. They also intend to develop students' understanding of the meaning of mathematical reasoning as well as of the topic under investigation. The latter of these intentions implies a shift of emphasis from *proving that* to *proving why* in school mathematics.

One suggestion for how to focus on *proving why* is to rely on generic arguments. They can be based on a *single-case key idea inductive argument* (Morris, 2007). As an example, consider the case of Larry, a grade 5 teacher whose class is working on perfect squares (Skott, in press). The students have previously made geometrical representations of square numbers with centicubes (cubes that may be assembled and used e.g. for teaching place value). The class has now made a table of the natural

numbers from 1 to 14 and their squares on the board. This leads to the observation that  $5^2 - 4^2 = 9 = 5 + 4$ . A single-case key idea inductive argument (not developed in the class) may build on the geometric representations used before. Placing two squares with side lengths 4 and 5 on top of each other with two pairs of sides aligned provides an explanation of the result and may be used to develop a generic argument that the difference between two consecutive perfect squares is the sum of their bases.

While it may alleviate some problems with R&P to engage students in investigating and conjecturing and focus on *proving why*, it may not sufficiently address others. In particular, students may become involved in the first two phases of the reasoning-and-proof cycle, but still rely on empirical or other justifications that do not qualify as mathematical. In Bieda's (2010) multiple case study experienced middle school teachers use a textbook that emphasizes R&P. In class, students produce conjectures in response to textbook tasks, but only in about half the cases do they provide some form of justification. Further, students' example-based justifications are accepted as much as their more general ones, and they had little opportunity to develop understandings of the specifics of mathematical reasoning and proof. A possible explanation, Bieda says, is that the teachers become involved in a reform agenda that prioritises "student-centred teaching", which requires them to play a relatively unobtrusive role in relation to the students' learning.

Similarly, Larry (cf. above) never capitalised on the students' conjecture that  $(n+1)^2 - n^2 = (n+1) + n$ , and sought to develop a mathematically valid justification, generic or otherwise. Also, the first author's longitudinal study of a teacher, Anna, suggests that her intention of supporting the development of students' proficiency with mathematical communication and R&P is often submerged by other concerns, e.g. not to jeopardise her relationship with the students (Skott, 2013). Consequently she accepts arguments and justifications that do not qualify as mathematical.

The studies mentioned above suggest that teachers find it difficult to capitalise on the R&P potential of situations that "arise naturally from students' work as they explore mathematical phenomena, examine particular cases, discuss alternative hypotheses, and generate conjectures" (A. J. Stylianides & Ball, 2008, p. 312). As a result of the difficulties, mathematical R&P may lose its content specificity. We suggest that PoP may be able to explain why, and we let these explanations inform our development initiatives on R&P in MTE.

## **THE PATTERNS-OF-PARTICIPATION FRAMEWORK AND RAPITE**

The PoP-framework adopts a participatory approach to human functioning, drawing on social practice theory (e.g. Holland, Skinner, Lachicotte Jr, & Cain, 1998; Lave, 1997; Wenger, 1998) and on Sfard's theory of commognition (Sfard, 2008). These frameworks focus, respectively, on emerging social processes (e.g. romance at a US university campus, cf. Holland & Eisenhart, 1990) and on well-structured cultural practices (e.g. mathematics, cf. Sfard, 2008). Rather than focusing on the practices

per se, however, PoP re-centres the individual and asks how a teacher's involvement in unfolding school and classroom events relates to and is transformed by her re-engagement in other past and present practices and discourses. We have found the I-me distinction in symbolic interactionism (Blumer, 1969; Mead, 1934) helpful for this purpose. It allows us to focus on how the teacher takes the attitude to herself of different individual and generalized others as classroom processes unfold.

If, for instance, a teacher seeks to develop a good mathematical argument with a group of students, who appear to be weak and vulnerable in the situation, she may simultaneously take the attitude to herself of colleagues who focus on creating trusting relationships with the students; of the school leadership or of parents, who emphasise students' performance on standardized tests; or of her teacher education programme that focuses on the use of manipulatives to facilitate student learning with understanding (Skott, 2013, 2015; Skott, Larsen, & Østergaard, 2011). The teacher's engagement with each of these social constellations – or others – may transform or subsume her involvement in the practice of mathematical R&P and for instance have her accept justifications that do not qualify as mathematical. PoP provides a perspective on if and how this is the case.

PoP has so far framed studies conducted “in the perspective of teacher education” (Krainer & Goffree, 1998). These studies are not on MTE, but develop understandings of teaching-learning practices in schools and may raise questions about MTE and inform decisions on how to address them. As indicated above, the results suggest that even when teachers engage students in elements of the R&P cycle, modes of justification may lose their subject specificity. To avoid this it seems that MTE needs to fulfil two requirements. First, it must be close to teaching-learning processes in schools, as R&P practices are otherwise too distant from classroom interaction for teachers to draw on them when teaching. This is in line with the suggestion to emphasise *proving why* using generic arguments (Rowland, 2002). Second, and in spite of that, MTE must be close to the disciplinary practice of R&P and include significant elements of *proving that* so as to make mathematical R&P a practice for teachers to draw on as they interact with their students and to limit the risk of classroom processes losing their subject specificity. The assumption of RaPiTE, then, is that MTE needs to avoid the two extremes of focusing either on academic mathematics or school mathematics, not by reducing the emphasis on either but by transforming both (Skott, in press).

To be “sufficiently close” to both school mathematics and academic mathematics we use tasks and conjectures that may be used in or developed from tasks used in school and take them beyond the school level. Examples include:

(1) *Does 8 always divide  $n^2-1$ , if  $n$  is an odd integer?* (This is from an interaction in Larry's grade 5, cf. the previous example on perfect squares (Skott, in press));

(2) Assume that you have a set of rods similar to Cuisenaire rods representing the positive integers from 1 to  $n$ . For what values of  $n$  can you make two “trains” of rods of equal length? Three trains?  $m$  trains?

## **THE PILOT STUDY**

The pilot study takes place at prestigious college in Denmark. The student teachers (from now on: teachers) have all performed fairly well in secondary school, and according to curricular documents they have worked with mathematical reasoning both in primary and secondary school. At the college, they need to specialize in Danish or mathematics. The research participants are a class of 31 prospective teachers for grades 4-9, who are among the 35 %, who specialize in mathematics.

### **Organisation and methods**

The pilot consists of two parts, a questionnaire and a short teaching-learning sequence on R&P in connection with the teachers’ first practicum. We do not expect the questionnaire and the observations of the teaching-learning sequence to shed light on relatively stable and context-independent mental constructs. Also, we do not assume any causal relation between responses to the questionnaire and teachers’ contributions to classroom practice. At best, the questionnaire allows us to understand how the teachers react discursively to R&P in a setting in which they are not challenged by other concerns that may emerge in classroom interaction. From a PoP perspective it is an empirical question, whether teachers orient themselves towards such a discourse as they engage with their students in the classroom. However, if teachers face significant problems with R&P in the questionnaire, we consider it unlikely that they engage proficiently with these processes when teaching.

At the beginning of the academic year the teachers fill in the questionnaire, which consists of open items on why they decided to go into teaching, why they chose to specialize in mathematics, and what their general experiences are with school mathematics. They are also asked about specific experiences with R&P (e.g. “Describe how you felt about reasoning and proofs in mathematics”) and to consider situations from school mathematics with an element of mathematical reasoning.

The second part of the pilot, the focus of the present paper, is the teaching-learning sequence on R&P, which is connected to the teachers’ first practicum. As part of their first course on mathematics at the college, they are re-introduced to R&P in a 12-lesson sequence, organised as two sessions of six 45-minute lessons. This sequence was not taught by the authors of the present paper, but the second and third authors planned it and developed the teaching-learning materials. The intentions and the contents were discussed in detail with the colleague, who taught the sequence.

In the sequence the teachers are introduced to different types of arguments (cf. G. J. Stylianides & Stylianides, 2009), which leads to discussions of why R&P is taught in school, of what to expect in terms of student learning, and of the relationship and

possible transition from empirical arguments to proofs. Also, the teachers watch and discuss a video of school students making and justifying conjectures about a number pattern in a sequence of geometric figures. This leads to discussions about the quality of the students' arguments and how students may be supported in developing them further. Subsequently, the teachers become involved in all three parts of the R&P cycle, for instance as they work on a version of the second task mentioned previously on making "trains" of equal length. As part of this they are to make geometrical or number theoretical justifications for their claims. They also discuss comments from school students, who have previously worked on the same task. One of these reads:

"If I am to make two trains of equal length the sum must be even. If the sum is odd, I would have one left over. If the sum is even there could be other problems [...]. We do not know if it is sufficient that the sum is even."

After these sessions on mathematical R&P, the teachers form eight groups of three or four, each group going on a two-week practicum in a middle or lower secondary school. Before and during their practicum the students are to (1) plan for their students' involvement in R&P; (2) video record each other's teaching; and (3) select one video clip from the practicum in which the students are particularly involved in R&P. After the practicum, the teachers discuss the video clips and the inherent potentials for and problems with R&P in a whole-day session. Below we focus on the teachers' response to the last requirement and on the subsequent discussion.

The sequence on R&P before and after the practicum was video recorded and transcribed. Like the responses to the questionnaire, the transcripts were analysed with no pre-developed set of codes, using coding procedures inspired by grounded theory (Charmaz, 2006). The initial coding and categorization of the data material was first done by authors 2 and 3 independently. The coding included word-by-word, line-by-line, incident-to-incident and in Vivo coding (Charmaz, 2006). Memo-writing was used increase the level of abstraction. Subsequently codes and categories were compared and discussed among all authors and inconsistencies were resolved.

The analysis resulted in categories on (1) teachers' reasons for selecting the specific video clips; (2) the character of R&P in explicit discussions of these processes; (3) student learning and its possible relation to R&P; (4) the significance of R&P in school mathematics; (5) "blackboard-talk", that is, whole-class teaching as it relates to student learning in general and to R&P in particular.

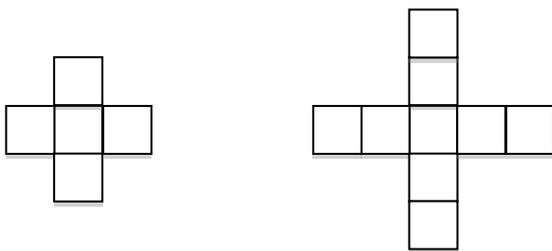
## **Results**

The results of the questionnaire support previous findings that many teachers have difficulties with deciding what a valid mathematical argument is. This is the case also for a large proportion of the teachers, who in the context of the questionnaire claim to be good at mathematics and to like engaging in mathematical R&P.

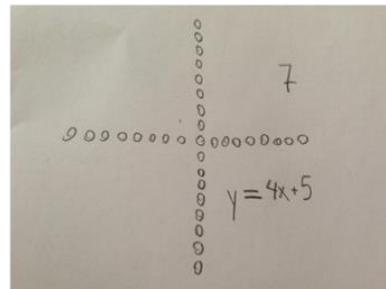
In the observations from the college classroom, the teachers face considerable problems arguing how or why the video clips they selected from their practicum is

related to R&P. Four of the groups do not provide a coherent explanation for why they selected the episode, and three of the other groups select the clips for reasons that are unrelated to mathematical reasoning. The last group claims that their clip is on reasoning, but it shows students making number stories for tasks on fractions.

Looking at the clips themselves, rather than at the teachers' reasons for selecting them, three have no connection to mathematical reasoning (e.g. the teacher presents the solution to a procedural task on the board). The other episodes have some potential for student involvement in R&P, but the teachers do not emphasise aspects of R&P in the discussion in the classroom.



**Figure 1**



**Figure 2**

One example with some potential for involving the students in the phases of the R&P cycle concerns the pattern in the number of squares in a sequence of figures (fig. 1). The students, who are in grade 6, use cookies to represent the squares. The video clip shows two students, who have 33 cookies. They have written “7” and “ $y=4x+5$ ” and made the drawing in fig. 2. However, they have trouble linking the equation to the geometric representation, and as the teacher joins them, the emphasis of the discussion is on the meaning of  $x$  and the length of the arms of the cross. This becomes a major concern also in the discussion at the college, which also revolves around the work of other students, who have written the same equation, but begun to solve it for different values of  $y$ , and around more general pedagogical issues such as how much support students should have in a situation like this. At no point does it become an issue if and how the episode could become the starting point for formulating a conjecture in the form of statement that could be verified. In this sense, mathematical justifications in the form of proving never become an issue.

Another example with R&P-potential concerns finding the point equidistant from the vertices of a triangle. In the video clip, a school mentor, the teacher normally teaching the class, unintentionally shows the students an incorrect procedure for constructing perpendicular bisectors. The students use the incorrect procedure but having measured the distances on their drawing they realise that something is wrong. They then shift their attention to the question of how to draw a perpendicular bisector and pay no attention to why it may help them solve the initial problem. In the discussion at the college the teachers discuss the episode, focusing on what they

describe as lack of conceptual understanding on the part of the students and on the use of whole-class instruction in general. It does not become an issue if and how the episode may become a starting point for an exploration of the problem, for formulating a conjecture on properties of perpendicular bisectors, let alone developing justifications for such conjectures.

## DISCUSSION AND CONCLUSIONS

The pilot confirms that teachers often face problems with R&P. The questionnaire establishes a setting remote from the classroom and the results do not in and by themselves indicate how the teachers react to similar questions when teaching. However, a PoP perspective suggests that the risk of not engaging sufficiently with R&P is greater in classrooms with many other pressing concerns beyond the quality of a mathematical argument. Further, our observations indicate that the teachers face problems identifying classroom situations with R&P, and in episodes with some potential for mathematical justification, modes of argumentation lose their subject specificity and conjectures are not subjected to mathematical verification.

In the cookie-episode the students engage in the important task of finding and generalising a pattern. Their difficulties may have been alleviated by moving the arms of the cross into a rectangular array with four columns and the centre cookie left over. This may be used as a single-case key idea inductive argument that builds only on rectangular representations of multiplication, and which may be turned into a generic argument that shows why the equation is right for all  $n$ . For this to happen, the teacher needs sufficient experiences with proving why in school contexts for it to become a mathematical practice (s)he can draw on in the interaction. It is of obvious importance that programmes for teacher education provide such experiences.

In other situations students' suggestions do not lend themselves as easily to generic arguments that *prove why*. This is the case for instance with the conjecture from Larry's classroom that if  $n$  is odd, 8 divides  $n^2-1$ . However, straightforward algebraic arguments and proof by induction may be used to prove the conjecture. However, if teachers are not sufficiently familiar with such arguments they have no alternative but to rely on the empirical ones used by the students.

From a PoP perspective these examples indicate that teachers need significant experiences with both *proving that* and *proving why*, if they are to support R&P activities in the classroom. The emphasis on *proving that* does not advocate a return in MTE to standard university courses with no relation to classroom practice; the mathematical practices involved are in that case too remote from school mathematics for teachers to draw on them in classroom interaction. However, using examples from school mathematics to develop means of *proving that*, including the much criticised proof by induction (Rowland, 2002), is necessary if teachers are to develop sufficient proficiency with dealing with all aspects of the reasoning and proof cycle in the classroom and capitalise on the potentials of their students' conjectures.

The pilot study examined the feasibility of proposals for MTE that are based on research with practising teachers. Our conjecture is that MTE needs to be close to both school and academic mathematics for teachers to link mathematical proficiency to instruction. In the case of R&P this means drawing on genuinely mathematical modes of justification in the classroom. The pilot supports the conjecture.

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